Math 4550 - Homework # 3 - Direct products

Part 1 - Computations

1. Do the following calculations in the group $\mathbb{Z}_2 \times \mathbb{Z}_3$.
   (a) List all of the elements in $\mathbb{Z}_2 \times \mathbb{Z}_3$.
   (b) Calculate $(\overline{1}, \overline{1}) + (\overline{1}, \overline{1})$.
   (c) Find the inverse of $(\overline{1}, \overline{1})$.
   (d) Calculate the order of $(\overline{0}, \overline{2})$.
   (e) Show that $\mathbb{Z}_2 \times \mathbb{Z}_3$ is cyclic by showing that $(\overline{1}, \overline{1})$ is a generator.
2. Do the following calculations in the group $\mathbb{Z}_4 \times \mathbb{Z}_{12}$.
   (a) Calculate $(\overline{2}, \overline{8}) + (\overline{3}, \overline{7})$.
   (b) Calculate $(\overline{3}, \overline{5}) + (\overline{1}, \overline{11})$.
   (c) Find the inverse of $(\overline{3}, \overline{5})$.
   (d) Calculate the order of $(\overline{2}, \overline{3})$ and the elements of the subgroup $\langle(\overline{2}, \overline{3})\rangle$.
3. Do the following calculations in the group $\mathbb{Z}_3 \times \mathbb{D}_6$.
   (a) Calculate $(\overline{2}, sr)(\overline{2}, sr^2)$.
   (b) Calculate $(\overline{1}, r^2)(\overline{0}, sr)$.
   (c) Find the inverse of $(\overline{1}, r^2)$.
   (d) Find the order of $(\overline{1}, r)$.

Part 2 - Proofs

4. List the elements of $\mathbb{Z}_2 \times \mathbb{Z}_4$. Then show that
$$H = \{(\overline{0}, \overline{0}), (\overline{0}, \overline{2}), (\overline{1}, \overline{0}), (\overline{1}, \overline{2})\}$$
   is a non-cyclic subgroup of $\mathbb{Z}_2 \times \mathbb{Z}_4$.
5. Let G_1 and G_2 both be abelian groups. Prove that $G_1 \times G_2$ is abelian.
6. Let G be a group. Prove that if $G \times G$ is cyclic, then G is cyclic.